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COMMENT

Some aspects of the boson–fermion (in)equivalence: a remark on the paper by Hudson and Parthasarathy

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Abstract. We link the recently proposed unification of the boson and fermion stochastic calculus with the general problem of boson-fermion equivalence (duality, reciprocity, etc) for quantum fields. Even if via the Fock construction the common Fock space for bosons and fermions can be introduced, it still does not allow for the unrestricted boson-fermion equivalence for field theory models. All local fermion field theory models thus have boson equivalents (violating the weak local commutativity condition for space dimension three). The reverse statement is not valid: not all boson models admit a pure fermion reconstruction.

1. Fermion fields in the boson Fock space according to Hudson and Parthasarathy

The quantum analogue of the theory of stochastic processes and stochastic differential equations is the theory of stochastic integrals and differentials with respect to basic operator processes. For its construction Fock representations of the CCR [1] and CAR [2-4] algebra were used, thus resulting in the boson and fermion stochastic calculus respectively.

In a recent paper [5] fermion annihilation and creation processes were explicitly realised in the boson Fock space as functions of the corresponding boson processes. The inverse construction of boson processes from the fermion ones is realisable as well, thus leading to the identification of the boson and fermion Fock space.

We denote by $\mathscr{H}$ the boson Fock space over the Hilbert space $h = L^2(R_+)$, and the representation of the CCR algebra is generated by operators:

$$\begin{aligned} &[b(s), b^*(s')]_- \equiv \delta(s-s') \\ &[b(s), b(s')]_- = 0 = [b^*(s), b^*(s')]_- \qquad (1.1) \\ &b(s)\psi_0 = 0 \qquad \forall s \in R,\ \psi_0 \in \mathscr{H}. \end{aligned}$$

The differential form of the boson annihilation process is $\mathrm{d}B(s) = b(s)\,\mathrm{d}s$ and upon introducing the appropriate reflection process $J = J(s)$, $s \in R_+$ the fermion processes can be introduced in $\mathscr{H}$ such that

$$\mathrm{d}F^{\#} = J\,\mathrm{d}B^{\#} \qquad \mathrm{d}B^{\#} = J\,\mathrm{d}F^{\#} \qquad (1.2)$$

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where

$$
\begin{aligned}
&\mathrm{d}F(s) = a(s)\,\mathrm{d}s \\
&[a(s), a^*(s')]_+ = \delta(s-s') \\
&[a(s), a(s')]_+ = 0 = [a^*(s), a^*(s')]_+ \\
&a(s)\psi_0 = 0 \qquad \forall s \in R,\ \psi_0 \in \mathcal{H}.
\end{aligned}
\tag{1.3}
$$

Let $\Delta(n, t)$ denote the set $\{(s_1, \ldots, s_n) \in [0, t]^n : 0 \leqslant s_1 < s_2 < \ldots < s_n \leqslant t\}$. Lemma 4.1 in [5] states that boson and fermion processes of strength ϕ, $\phi \in L^2_{\mathrm{loc}}(R_+)$, are related by the following formulae to their differential versions:

$$
\begin{aligned}
F^*_{\phi_1}(t) \ldots F^*_{\phi_n}(t)\psi_0 &= \int_{\Delta(n,t)} \det(\phi_i(s_j))\, \mathrm{d}B^*(s_1) \ldots \mathrm{d}B^*(s_n)\psi_0 \\
B^*_{\phi_1}(t) \ldots B^*_{\phi_n}(t)\psi_0 &= \int_{\Delta(n,t)} \mathrm{per}(\phi_i(s_j))\, \mathrm{d}F^*(s_1) \ldots \mathrm{d}F^*(s_n)\psi_0
\end{aligned}
\tag{1.4}
$$

where

$$
\begin{aligned}
\mathrm{per}\, C &= \sum_{\pi \in s_n} \prod_{i=1}^{n} C_{i\pi(i)} \\
\det C &= \sum_{\pi \in s_n} (-1)^{\pi} \prod_{i=1}^{n} C_{i\pi(i)}
\end{aligned}
\tag{1.5}
$$

for the $n \times n$ matrix C.

The stochastic integrals in (1.4) can be converted to the Lebesgue integrals, so that the familiar field theoretic expressions for Fock space vectors arise:

$$
\begin{aligned}
F^*_{\phi_1}(t) \ldots F^*_{\phi_n}(t)\psi_0 &= \int_{\Delta(n,t)} \det(\phi_i(s_j)) b^*(s_1) \ldots b^*(s_n)\, \mathrm{d}s_1 \ldots \mathrm{d}s_n \psi_0 \\
B^*_{\phi_1}(t) \ldots B^*_{\phi_n}(t)\psi_0 &= \int_{\Delta(n,t)} \mathrm{per}(\phi_i(s_j)) a^*(s_1) \ldots a^*(s_n)\, \mathrm{d}s_1 \ldots \mathrm{d}s_n \psi_0.
\end{aligned}
\tag{1.6}
$$

A straightforward consequence of (1.6) is theorem 4.3 of [5] which identifies the fermion and boson Fock space, and allows for the conclusion that the existence of unitary stochastic evolutions driven by fermion and gauge noise is thereby reduced to an equivalent boson problem.

2. Representations of the CAR generated by representations of the CCR in Fock space: boson–fermion duality or non-duality?

Since in the above the CAR and CCR algebra generators in the boson Fock space have a common cyclic (vacuum) vector, the construction of [5] automatically falls into the framework of [6, 7] which provides a universal solution to the problem of embedding the CAR algebra in the (bicommutant of) CCR algebra on the level of Fock representations. It is thus also connected with the idea of the boson-fermion equivalence for field theory models: the two independent and, in fact, inequivalent lines of research should be mentioned here, that following Skyrme [8–11] and that arising from [6, 7] and continued in [12–16].

In connection with the Fock space notion let us emphasise that commonly the same name is attributed to two *a priori* distinct spaces: the Hilbert space of Fock vectors of the form (1.6) which we denote $\mathscr{H}$ and the Hilbert space of sequences of n-point functions $\mathscr{F}$. These sequences stand for coordinates of Fock space vectors in the occupation number basis. By virtue of the above fermion and boson Fock space unification, the choice of the boson basis would give rise to symmetric functions, while that of the fermion basis would give rise to antisymmetric functions. However they are merely different representatives of the same Fock space vector.

At this point the general study of [6, 7] intervenes. Once a Fock representation of the CCR algebra over K (in general, $K = L^2(R^N)$) is given, it automatically induces a Fock representation of the CAR algebra in the boson Fock space, which (we cite the main theorem of [7]) 'acts irreducibly on the following subspace of $\mathscr{F} = \mathscr{F}_B : \overset{1}{\mathscr{F}}_B = \bigoplus_{n=0}^{\infty} E_n^2 S_n K^{\otimes n}$', $\mathscr{F}_B = \bigoplus_{n=0}^{\infty} S_n K^{\otimes n}$, $K^0 = C$. Here we have an orthogonal decomposition $\mathscr{F}_B = \overset{1}{\mathscr{F}}_B \oplus \overset{2}{\mathscr{F}}_B$, S_n denotes the symmetrisation operator in the nth tensor product space $K^{\otimes n}$, E_n^2 is a projection in $K^{\otimes n}$, and its square root E_n has the property to convert the antisymmetric functions into the symmetric ones:

$$E_n(A_n K^{\otimes n}) = E^2(S_n K^{\otimes n}) \subset S_n K^{\otimes n} \tag{2.1}$$

and conversely, provided the symmetric function gets no contribution from $(1 - E_n^2) S_n K^{\otimes n}$.

It means that in terms of (Fock) function sequences, fermion Fock space can be identified with a proper subspace of boson Fock space having a non-trivial orthogonal complement $\overset{2}{\mathscr{F}}_B$. This complement makes a real distinction between the Bose and Fermi cases, unless it is trivial (it happens due to the special nature of the test function space and some peculiarities of the construction in [5]). It is the main purpose of this comment to reveal the role of this complement when proceeding to the study of concrete field theory models.

Let us make the following choice of $K = L^2(R_1)$ and let the integral kernel of E_n be given as follows [17]:

$$E_n(s_1, \ldots, s_n; t_1, \ldots, t_n) = \sigma(s_1, \ldots, s_n)\delta(s_1 - t_1) \ldots \delta(s_n - t_n) \tag{2.2}$$

where $\delta(s - t)$ symbolises the Dirac delta, while $\sigma(s_1, \ldots, s_n)$ is the Friedrichs-Klauder [6] totally antisymmetric (sign) symbol:

$$\sigma(s_{\pi(1)}, \ldots, s_{\pi(n)}) = (-1)^{\pi} \qquad s_i \neq s_j \tag{2.3}$$

where π denotes a permutation of indices. If for any pair of labels we have not satisfied $s_i \neq s_j$ the symbol σ equals 0. Upon exploiting the form (2.2) of E_n the following expression arises for the n-particle fermion vector in the boson Fock space:

$$\begin{aligned}
&a(f_1)^* \ldots a(f_n)^* \psi_0 = \int ds_1 \ldots \int ds_n f_1(s_1) \ldots f_n(s_n) \sigma(s_1, \ldots, s_n) b^*(s_1) \ldots b^*(s_n) \psi_0 \\
&b(f_1)^* \ldots b(f_n)^* \psi_0 = \int ds_1 \ldots \int ds_n f_1(s_1) \ldots f_n(s_n) b^*(s_1) \ldots b^*(s_n) \psi_0 \\
&a(f) = a(f, t) = \int_0^t ds\, \bar{f}(s) a(s) \\
&[a(f), a(g)^*]_+ = \int_0^t \bar{f}(s) g(s)\, ds \qquad [a(f), a(g)]_+ = 0
\end{aligned} \tag{2.4}$$

$$[b(f), b(g)^*]_- \equiv \int_0^t \bar{f}(s)g(s)\,\mathrm{d}s \qquad [b(f), b(g)]_- = 0$$

$$a(f)\psi_0 = 0 = b(f)\psi_0 \qquad \forall f \in L^2[0,t] = L^2_{\mathrm{loc}}(R_+).$$

All integrations are carried out with respect to the Lebesgue measure. Hence if we specialise considerations to the $n=2$ case (extensions to arbitrary n are straightforward), we arrive at

$$\begin{aligned} a(f_1)^*a(f_2)^*\psi_0 &= \int \mathrm{d}s_1 \int \mathrm{d}s_2\, f_1(s_1)f_2(s_2)\sigma(s_1, s_2)b^*(s_1)b^*(s_2)\psi_0 \\ &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2\, f_1(s_1)f_2(s_2)b^*(s_1)b^*(s_2)\psi_0 \\ &\quad + \int_{s_2<s_1} \mathrm{d}s_1\, \mathrm{d}s_2\, f_1(s_1)f_2(s_2)\sigma(s_1, s_2)b^*(s_1)b^*(s_2)\psi_0. \end{aligned} \tag{2.5}$$

The contribution $\int_{s_1=s_2} \mathrm{d}s_1\, \mathrm{d}s_2$ from the set of Lebesgue measure zero has been omitted (we made use of such omission possibilities in our model studies of [13, 17, 19]). Let us mention that Klauder was the first [18] to publicly state that, due to the Lebesgue measure involved, the fermion and boson transition amplitudes can be identified.

In the second term we have $s_2 < s_1$ and hence $\sigma(s_1, s_2) = -1$. If now we change the variables $s_1 \leftrightarrow s_2 \Rightarrow \int_{s_2<s_1} \to \int_{s_1<s_2}$ and consequently $f_1(s_1)f_2(s_2) \to f_1(s_2)f_2(s_1)$ we find

$$\begin{aligned} a(f_1)^*a(f_2)^*\psi_0 &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2 (f_1(s_1)f_2(s_2) - f_1(s_2)f_2(s_1))b^*(s_1)b^*(s_2)\psi_0 \\ &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2\, \det(f_i(s_j))b^*(s_1)b^*(s_2)\psi_0 \end{aligned} \tag{2.6}$$

which is precisely one of the Hudson-Parthasarathy formulae. The other one arises due to

$$\begin{aligned} b(f_1)^*b(f_2)^*\psi_0 &= \int \mathrm{d}s_1 \int \mathrm{d}s_2\, f_1(s_1)f_2(s_2)b^*(s_1)b^*(s_2)\psi_0 \\ &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2 (f_1(s_1)f_2(s_2) + f_1(s_2)f_2(s_1))b^*(s_1)b^*(s_2)\psi_0 \\ &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2 (f_1(s_1)f_2(s_2) + f_1(s_2)f_2(s_1))\sigma(s_1, s_2)[\sigma(s_1, s_2)b^*(s_1)b^*(s_2)]\psi_0 \\ &= \int_{s_1<s_2} \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{per}(f_i(s_j))a^*(s_1)a^*(s_2)\psi_0 \end{aligned} \tag{2.7}$$

provided we observe that $\sigma(s_1, s_2) = 1$ because of $s_1 < s_2$ and omit the contribution from sets of Lebesgue measure zero. We have thus demonstrated that the unification of the boson and fermion stochastic calculus by Hudson and Parthasarathy involves what is in effect the special case of our CAR = CAR(CCR) construction [6, 7].

The use of Lebesgue measure guarantees a complete identification of boson and fermion Fock spaces not only when $K = L^2(R_1)$ but also in the case of $K = \bigoplus_1^M L^2(R^N)$ as well. However, one must keep in mind that Fock space vectors determine the corresponding function sequences up to contributions from sets of Lebesgue measure zero. If these contributions are not taken into account while passing to the study of field theory models, quite serious problems are apparently encountered. Recall that $\mathscr{F}_B = \bigoplus_{n=0}^{\infty} S_n K^{\otimes n} = \overset{1}{\mathscr{F}}_B \oplus \overset{2}{\mathscr{F}}_B$ and the $\overset{2}{\mathscr{F}}_B = \bigoplus_{n=0}^{\infty} (1 - E_n^2) S_n K^{\otimes n}$ piece is cancelled in the Fock construction which maps function sequences into vectors:

$$\{f_n(s_1, \ldots, s_n)\} \to |f\rangle = \sum_n \int ds_1 \ldots \int ds_n f_n(s_1, \ldots, s_n) b^*(s_1) \ldots b^*(s_n) \psi_0. \tag{2.8}$$

However, what happens if we act upon such vectors by operators? To clarify the situation we shall discuss one paradigm example of the concrete field theory model we have studied before [17] in search of mechanisms of the fermion-boson reciprocity: the non-linear Schrödinger field in $(1+1)$ dimensions, with a repulsive potential (now a configuration space variable appears instead of the previously used time variable $s \in R$). The spectral solution for the Hamiltonian:

$$\begin{aligned} &H = -\tfrac{1}{2} \int dx\, \phi_x^* \phi_x + \tfrac{1}{2} c \int dx\, \phi^*(x)^2 \phi(x)^2 \\ &[\phi(x), \phi(y)^*]_- \equiv \delta(x-y) \qquad [\phi(x), \phi(y)]_- = 0 \\ &\phi(x)\psi_0 = 0 \qquad \forall x \in R \end{aligned} \tag{2.9}$$

is looked for in the Fock space $\phi(x)\psi_0 = 0\ \forall x \in R$. According to our previous analysis (we omit the contributions from sets of Lebesgue measure zero) the n-particle vector acquires the following form:

$$\phi(f_1)^* \ldots \phi(f_n)^* \psi_0 = \int_{x_1 < \ldots < x_n} dx_1 \ldots \int dx_n\, \mathrm{per}(f_i(x_j)) \phi^*(x_1) \ldots \phi^*(x_n) \psi_0. \tag{2.10}$$

If one would perform H of (2.9) naively upon (2.10), we would realise that on all Fock space vectors (vary n) the interaction term identically vanishes, thus reducing the non-trivial model to the free field case which is known to arise in the $c = 0$ limit (free boson) or $c \to \infty$ (free fermion).

By virtue of the boson and fermion Fock space unification these two free field models are equivalent (compare, e.g., [16]) since the free boson Hamiltonian acts invariantly in any domain $\mathscr{D} \subset \overset{1}{\mathscr{F}}_B$.

It is not the case when $0 < c < \infty$, since then the proper domain for H is $\mathscr{F}_B = \overset{1}{\mathscr{F}}_B \oplus \overset{2}{\mathscr{F}}_B$:

$$\begin{aligned} |f\rangle &= \int dx_1 \ldots \int dx_n f(x_1, \ldots, x_n) \phi^*(x_1) \ldots \phi^*(x_n) \psi_0 \\ H|f\rangle &= \int dx_1 \ldots \int dx_n \left\{ \left(-\frac{1}{2} \sum_{j=1}^{n} \nabla_j^2 + \frac{c}{2} \sum_{i \neq j} \delta(x_i - x_j) \right) f(x_1, \ldots, x_n) \right\} \\ &\quad \times \phi^*(x_1) \ldots \phi^*(x_n) \psi_0 \end{aligned} \tag{2.11}$$

shows that for the study of spectral properties the solutions of the eigenvalue problem

$$(H_n f)(x_1, \ldots, x_n) = E f(x_1, \ldots, x_n) \tag{2.12}$$

are necessary and the many-body Hamiltonian H_n non-trivially mixes $\overset{1}{\mathscr{F}}_B$ and $\overset{2}{\mathscr{F}}_B$ in $\mathscr{F}_B$.

At this point let us recall that the standard (Gel'fand-Neumark-Segal) reconstruction procedure by using the CAR algebra would give rise to the $\overset{1}{\mathscr{F}}_{B}$ piece only ($\overset{1}{\mathscr{F}}_{B}$ is closed). If we adopt the CCR algebra reconstruction we would arrive at $\mathscr{F}_B = \overset{1}{\mathscr{F}}_{B} \oplus \overset{2}{\mathscr{F}}_{B}$ and obviously $\overset{2}{\mathscr{F}}_{B}$ is beyond the reach of the previous (CAR) procedure. It clearly demonstrates that not all boson models allow for the pure fermion reconstruction (the reverse is always true); even if we pass from $\mathscr{F}_B$ to the Fock space $\mathscr{H}$.

It is impossible unless the boson Hamiltonian acts invariantly in $\overset{1}{\mathscr{F}}_{B}$, i.e. commutes with the projection on this proper subspace of $\mathscr{F}_B = \overset{1}{\mathscr{F}}_{B} \oplus \overset{2}{\mathscr{F}}_{B}$. Then contributions from $\overset{2}{\mathscr{F}}_{B}$ can be eliminated as irrelevant and the boson-fermion duality makes sense, albeit on the level of the relativistic field theory, the requirement of weak local anticommutativity for fermions would necessarily lead to the violation of weak local commutativity for the related (dual) bosons.

Remark 1. The situation in continuum is entirely different from this for the lattice systems (even infinite), since the lattice analogue of the construction [6, 7], see e.g. [13], involves a decreasing family of projections in the Fock space for the Bose system and its proper subspace of Fermi states is always accompanied by the non-trivial orthogonal complement, whose contributions can never be neglected: it is a subspace of the Fock space $\mathscr{H}$ itself. Consequently there is no way at all to give a fermion reconstruction of the Bose system (the reverse is always true) unless a restriction to the appropriate subspace is imposed, see also [20], or irreducibility of representations abandoned.

Remark 2. For each Fermi system an equivalent Bose one can be found (irrespective of what is the spacetime dimension adopted). By virtue of the fact that the total set of exponential vectors (coherent states) spans the domain for equivalent Bose and Fermi systems, the standard tree approximation methods [13] allow us to attribute an unambiguous meaning to the notion of the classical relative for the Fermi system, which is a c-number (commuting function ring) field theory, unpleasant news for those field theory pragmatists who seriously claim that the classical level for Fermi fields is Grassmann algebra-valued.

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I also wish to make a special dedication to John Klauder, whose ideas [18] have seemed to materialise: genuine classical (c-number commuting function ring) spinors *do* have a meaning while a construction of Fermi models is attempted.

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